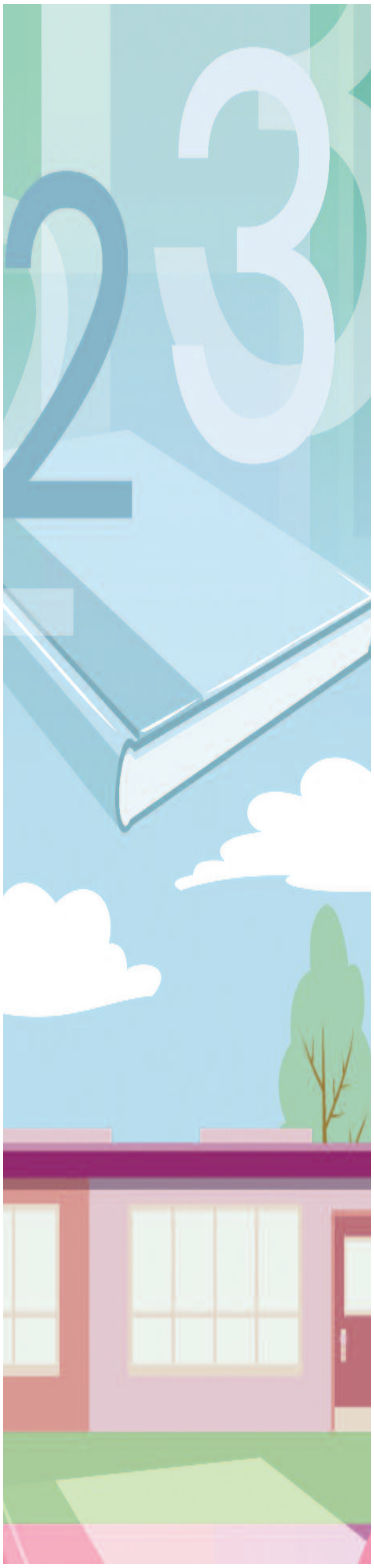

The Components of Number Sense

An Instructional Model for Teachers

Valerie N. Faulkner

In recent years much attention has been placed on the relatively poor math performance of students in the United States (Gonzalez et al., 2004; Lemke et al., 2004; National Center for Education Statistics, 1999; National Research Council, 2001). Increased attention has also been paid to the struggling learner and mathematics. This includes issues regarding assessment (Gersten, Clarke, & Jordan, 2007); low-performing students in reform-based classrooms (Baxter, Woodward, & Olson, 2001); and general recommendations for the struggling student by the National Math Panel (Gersten et al., 2008).

The mathematical knowledge of teachers has also been investigated, and student success has been tied to the subtle factors of teacher implementation choices regarding problem sets, questioning techniques, and math connections (Hiebert & Stigler, 2000; Hill, Rowan, & Ball, 2005; Stigler & Hiebert, 2004). Strong teacher implementation choices appear to be influenced by teacher knowledge and flexibility with the mathematics being taught. Furthermore, it has been demonstrated qualitatively that elementary teachers in the United States tend to lack a "profound understanding" of the fundamentals of the mathematics they teach (Ma, 1999).

Number Sense and Instructional Practice

At the heart of the recent focus on mathematics has been an increased emphasis on developing students' *number sense.* Ironically, although growing as a force in the education literature, number sense has not been clearly defined for teachers.

Teachers need specific support in understanding how to develop number sense in students, to guide their learning as they plan for and provide instruction (Ball & Cohen, 1996) and, ultimately, to ensure that they are spending time encouraging students to do the thinking that will improve number sense. A focus on content knowledge has been found to be an effective component of professional development for teachers (Garet, Porter, Desimone, Birman, & Suk Yoon, 2001; Hill et al., 2005), and teacher content knowledge in mathematics has an impact on student performance (Hill et al.). In our work with hundreds of teachers throughout our state, we have found it necessary to support teachers with a model for number sense development that, first and foremost, supports a deep understanding of the mathematics itself. Using this model as the framework for the North Carolina Math Foundations training, we have been able to show teacher knowledge growth as measured by the Learning

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Mathematics for Teaching (LMT) Measures developed at the University of Michigan.

Teachers are increasingly faced with standard course of study documents listing number sense as a goal of instruction (e.g., in Washington, Missouri, North Carolina). These standards tend to present number sense in a perfunctory fashion that does little to delineate for the teacher how students acquire that number sense. Even those who do research to develop our understanding of number sense continue to refer to the phrase "difficult to define but easy to recognize" (Gersten, Jordan, & Flojo, 2005). In 2001, Kalchman, Moss, and Case describe number sense as

> The characteristics of good number sense include: (a) fluency in estimating and judging magnitude, (b) ability to recognize unreasonable results, (c) flexibility when mentally computing, (d) ability to move among different representations and to use the most appropriate representations. (p. 2)

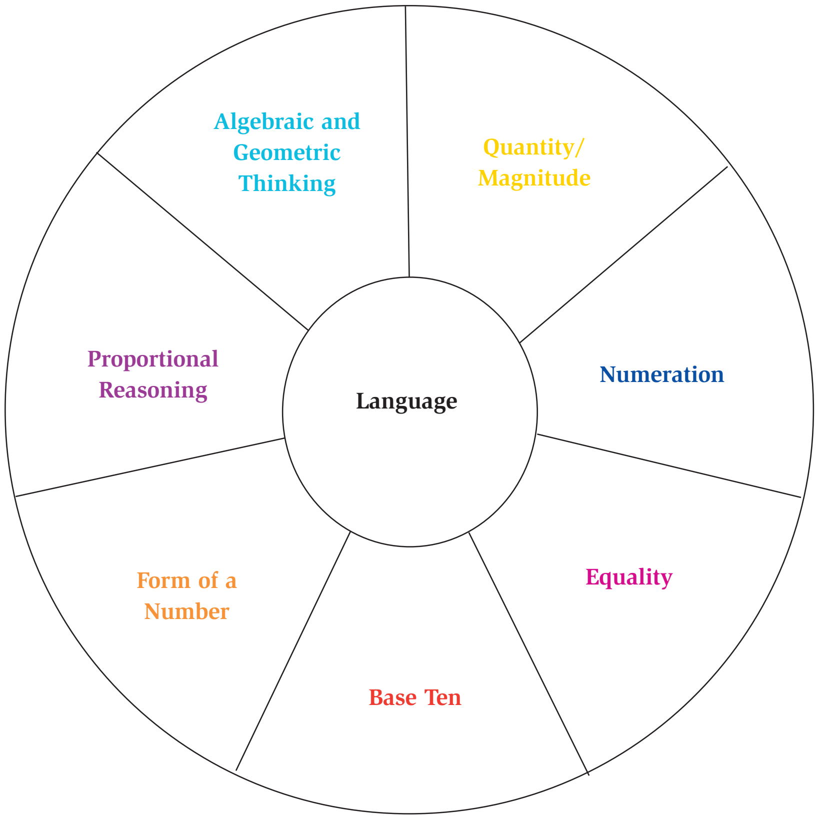
But for the teacher, the questions still remain: How do I get my pupils to gain these characteristics? What does this mean about how I should teach mathematics?

In other words, number sense is poorly outlined for the teaching community (if students can solve certain problems, then they have number sense) and is essentially defined in circular terms. This circular tendency perhaps reflects, unwittingly, a cultural vision of mathematical ability as something that is gifted to the individual rather than learned through specific patterns of habit and practice (Dehaene, 1997). Yet, mathematical ability probably falls more under the type of skill described through paradigms for developing expertise than through primarily native ability (Dehaene). This means that engaging with and practicing the right things will have an impact on mathematical understanding and performance. But what are the right things?

Although we have found that teachers strive to communicate mathematics effectively, they often struggle with identifying and emphasizing the critical mathematical structures for students. Our anecdotal experiences are virtually indistinguishable from the experiences of others who have documented teacher difficulties with mathematical content (Ball, 1990, 1992; Knuth, 2002; Ma, 1999; Post, 1991). These documentations describe teachers who generally endorse the importance of conceptual understanding and can often derive a correct answer, but are consistently unable to explain the mathematical logic behind an answer. We contend that this weakness in articulating a given mathematical structure reflects the teachers' learned understanding of the math. They understand mathematics as they were taught it: through procedures. We think of this as a fundamentally cultural issue and realize that we are asking teachers to break the chain of how they were taught.

What does this mean about how I should teach mathematics?

Figure 1. The Components of Number Sense



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The Components of Number Sense: Supporting Classroom Instruction

In order to support teachers' efforts to improve their mathematics instruction, we have devised a model for number sense. This model, The Components of Number Sense (created by C. Cain, M. Doggett, V. Faulkner, and C. Hale, 2007; see Figure 1 and box, "The Components of Number Sense: A Brief Outline"), represents discussions and connections that are to be made in

The Components of Number Sense: A Brief Outline

Valerie N. Faulkner

Quantity/Magnitude

Math is not "about numbers" it is "about quantity" (Griffin, 2003). Virtually all mathematical topics can be modeled for students using quantity as a core communicator. Imagine the following: Algebra taught through models of shopping with average price for slope, gas or bus fare as a constant; division of fractions taught through considerations of portion size; word problems taught through issues of quantity rather than decontextualized "key words." Quantity is the real topic of mathematics and students can be taught that they can model the world through mathematics.

Numeration

Numeration is a critical skill embedded in mathematical expression; it is essentially a code to be cracked. In order to become fluent in the language of mathematics and to develop a number sense, students must understand the idea that we group at the rate of 10 in our numeration system. Teachers are asked to be more conscious of the numeration system itself in their discussions with students. For instance, encourage students to name numbers in expanded form (23 becomes 2 tens and 3 ones), and to justify regrouping through an explanation of composing and decomposing numbers at the rate of 10 (Ma, 1999).

Equality

Equality is a powerful tool in mathematics. Liping Ma (1999) recalls a teacher of hers who called equality "the soul of mathematics." There are two common problems that arise as students develop an understanding of equality. One is thinking that equals means "the same as." It is important for the teacher to note that equality does not mean the same as and to use more accurate language with students. Two trucks may be equal in weight to an elephant, but they certainly aren't the same as! We want students to understand, when they see, for instance, $X = Y$ that this means here are two things that are not the same exactly, but they are equal in value. The second common problem is the idea that the equals sign is a directional signal. Perhaps because of a habit of our instruction (unsimplified term on the left, simplified term on the right) students see the equals sign to mean put your answer next. Work with students to develop this idea of setting two things equal to each other. (See the book, *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School,* by Carpenter, Franke, & Levi, 2003 for more ideas regarding lessons in equality.)

A corollary to this is the practice of manipulating terms through the identity principle, but not pointing out to students that really we are maintaining equality, so we are not changing the value of a term. For instance, when we tell students, when changing the form of a fraction so that we can add with like denominators, to "do the same thing to the top as you do to the bottom" we are merely presenting a procedure. A more meaningful presentation would be to say—

> We know that we can't just change what this fraction equals, I mean that would mess everything up. But we could multiply by 1, right? What if we multiply by 3/3—would that change the value? No—we would have a different form of the same value.

Base Ten

A consistent habit of practice utilizing base 10 would include using the terminology "power of 10" rather than "moving the decimal point over." Teachers should ensure that they are saying, for example "This is 6 times 10 times 10" rather than, "Add two zeros." In both cases, the former example creates a habitual connection to base 10, whereas the latter emphasizes a procedural habit that does not communicate base 10. Students need to think in powers of 10 so that, as the numbers they work with grow in magnitude, they are ready to assimilate them. Scientific notation is an important example. By developing an understanding that 600 is $6 \times 10 \times 10$ the student is better prepared to understand the value of 6.15×10^2.

Forms of a Number

This wedge is intrinsically tied to equality. We included it to support teachers in an important change in their language when discussing mathematics. Beginning with early understandings of number, we ask students to see that a set of •••• is simply a different form of presenting the symbol 4. Students who were 1 to 2 years behind in their math knowledge upon entry into kindergarten have been found to attain a level of achievement "indistinguishable from the normative group" after 2 years of mathematics instruction that teaches the ideas of magnitude and the number line specifically utilizing different forms of representing magnitude—sets, straight lines, circles, symbols, and so forth. (Griffin, 2003, 2004)

Consider the following mathematical topics: subtraction, fraction addition, and trigonometric proofs. These are all taught as novel concepts, yet by invoking the organizing idea of The Form of the Number, we begin to see what they have in common. Consistently connecting mathematical topics under this one umbrella will develop the students' sense for equality and their understanding that they, as mathematicians, can manipulate numbers yet maintain their values. This umbrella builds through the years and becomes a habit of thinking: the elementary school student asks, "Do I like the form this number is in?" when deciding whether to regroup, the middle schooler when adding or subtract-

ing fractions, and the high schooler when evaluating and manipulating values for trigonometric proofs. (The list goes on with regard to Form of a Number. Consider the following topics: simplifying expressions, combining like terms, converting mixed numbers to improper fractions, utilizing the distributive property, factoring, "FOILing." These topics can all begin with the question: "Do I like the form the number is in?")

Proportional Reasoning

Proportional reasoning involves the comparison of numbers within quantities as well as the comparison of numbers between quantities. "The essential characteristics of proportional reasoning involve the holistic reasoning between two rational expressions such as rate, ratios, quotients, and fractions" (Lesh, Post, & Behr, 1988). Proportional reasoning is a complex skill that has a direct correlation to success in higher mathematics. Inference and prediction are involved in the understanding of this concept as well as qualitative (non-numerical) and quantitative (numerical) comparisons. Rather than moving students quickly to the symbolic realm with proportions, students should instead be afforded the opportunity to develop diagram literacy (Deizmann & English, 2001). This habit of instruction would encourage actual proportional thinking, rather than the ability to fit numbers into an exchange formula.

Consider one of the most important proportions we have in mathematics: *Pi*. Although *Pi* (circumference/diameter) is *essentially* a proportion, it is taught and utilized in math classes almost exclusively as an irrational number or estimate thereof. A conscious instructional habit that includes proportional reasoning would find the teacher engaging students in questions based on the real proportional relationship of *Pi*: "If the diameter of a circle is 10, about what is the circumference? What if the circumference is 75, about what is the diameter? The radius?" This practice develops students' proportional reasoning, geometric thinking, and number sense in general. This opportunity stands in contrast to memorizing how to plug *Pi* into an equation to get an answer.

Algebraic and Geometric Thinking

This component is important particularly when we consider that this is where we want students to eventually be with their number sense and their mathematical understanding. It is important, though, to understand how early elementary mathematics is tied to algebraic and geometric thinking and not to only see these things as an end goal. Considering the earlier examples, we see how early understandings of equality affect algebraic thinking ($X = Y$), how proportions create deeper understanding of geometry (*pi* is a proportional tool, not just an irrational number), and how the components in general support mathematical number sense that enables students to handle the more demanding topics of algebra and geometry. Making these connections includes not only how arithmetic and mathematics are taught in the elementary school, but also how algebra and geometry are taught in the higher grades. Considering components of numbers sense, the algebra teacher might remember to explain slope with proportional pictures or diagrams before converting them to a symbolic form. Geometry teachers will be more sensitive to the idea that 1 may equal 1^3 but they are very different forms of the same value. Whereas one is a linear measurement, the other is a volume measurement. This is actually a tricky concept that must be "unpacked" and demands the attention of teachers so that students can develop their geometric thinking and their understanding of unit measures.

virtually every math lesson. This is not a progressive model wherein once one wedge is taught it is then seen as review material. Rather, each wedge is to be connected to each lesson throughout the curriculum. In this way, it is to be seen as a model that acts as a support for all mathematical discussions. By delineating these modular components of number sense, we hope to support teachers in developing number sense within their students by habitually making mathematical connections. This modular framework is analogous to the paradigm put forth by the neuropsychologist Stanislas Dehaene (1997) on the teacher's role in activating the "modules" of the brain:

> [S]chooling plays a crucial role not so much because it teaches children new arithmetic techniques, but also because it helps them draw links between the mechanics of calculation and its meaning. A good teacher is an alchemist who gives a fundamentally modular human brain the semblance of an interactive network. (p. 139)

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Estimation is frequently associated with number sense—but what allows a person to have a strong estimating mind? It is clear from Dehaene's work that there is not one spot that needs to be addressed for a skill such as estimation, and, yet, if one can estimate, one likely has some number sense. One who can estimate must, at the very least, draw on the concepts of quantity, magnitude, and proportional reasoning.

How the Components of Number Sense Affected One Middle School Math Teacher

Dr. Chris Cain

As teacher educators, we have prioritized providing teachers with a tool that will substantially support their efforts to change their daily habits of language and instruction. We feel strongly that research must be made accessible to teachers so that they can effect change in their classrooms. It is our contention that this Model for Number Sense does just that.

One such example came in the college class, Advanced Methods of Mathematics Instruction. One of the participants in the class was a middle school teacher who had returned for licensure in special education. Her attitude during the first few classes conveyed to me that she was just marking time in this special education class on mathematics because she already saw herself as proficient in mathematics instruction—after all she was a math education major. This class was merely a requirement to her. Nevertheless, I did my best to maintain her attention as I went through a lecture and discussion on the use of language as an introduction to the components of number sense.

When she walked in the classroom the following class I immediately noticed that she had a very different attitude. She spoke up, she asked questions, and she was open to new ideas. After class, I asked her what had prompted this change. She explained that yesterday in her classroom she was teaching the concept of 45% of 73. This was a lesson she had taught many times. She fully anticipated the one question that would come when she explained to the class that to change the percent into a decimal number you moved the decimal two places to the left—students would say "but why?" When the question came, as predicted, she gave her usual answer, "You just do." With this in mind she stopped the lesson and took ones, tens, and one hundreds blocks out of the closet and began modeling for the students the "why," utilizing the components of number sense. She placed two sets of hundreds blocks on the overhead, the two sets were divided by a line. Above one set she wrote 45% and beside the other she wrote .45. She needed to make the numeration system more clear to her students, so she spoke to the class about equality and then asked students to tell her how these two forms of a number are equal. The class had a very hard time explaining the reason why the two forms of the number were equal.

Next, she had asked the class to use the blocks to show her 45%. She asked, "This is 45% of what?"; the class just looked at her. She explained that cent means 100 as in century and, therefore, percent means per 100. They were then able to articulate that 45% must be 45 out of 100. Then she had a student come up to demonstrate this with the blocks. The student pulled down 45 out of the one hundred blocks.

Then she asked the class to explain to her what .45 is. They replied forty-five hundredths. When she asked what forty-five hundredths is, they could not explain. She explained to the class that 1 whole had been broken into 100 parts and .45 meant 45 of those little 1/100th parts. She then asked a student to come up and show her 45 out of the 1 or .45. The student walked up and pulled down 45 blocks.

She told me she had seen light bulbs come on across the room. They then discussed as a class why 45% can be seen as a different form of the decimal 0.45. She said it was one of the best math classes she had ever taught. She said her students truly understood and that she and her students were having their first real math conversation.

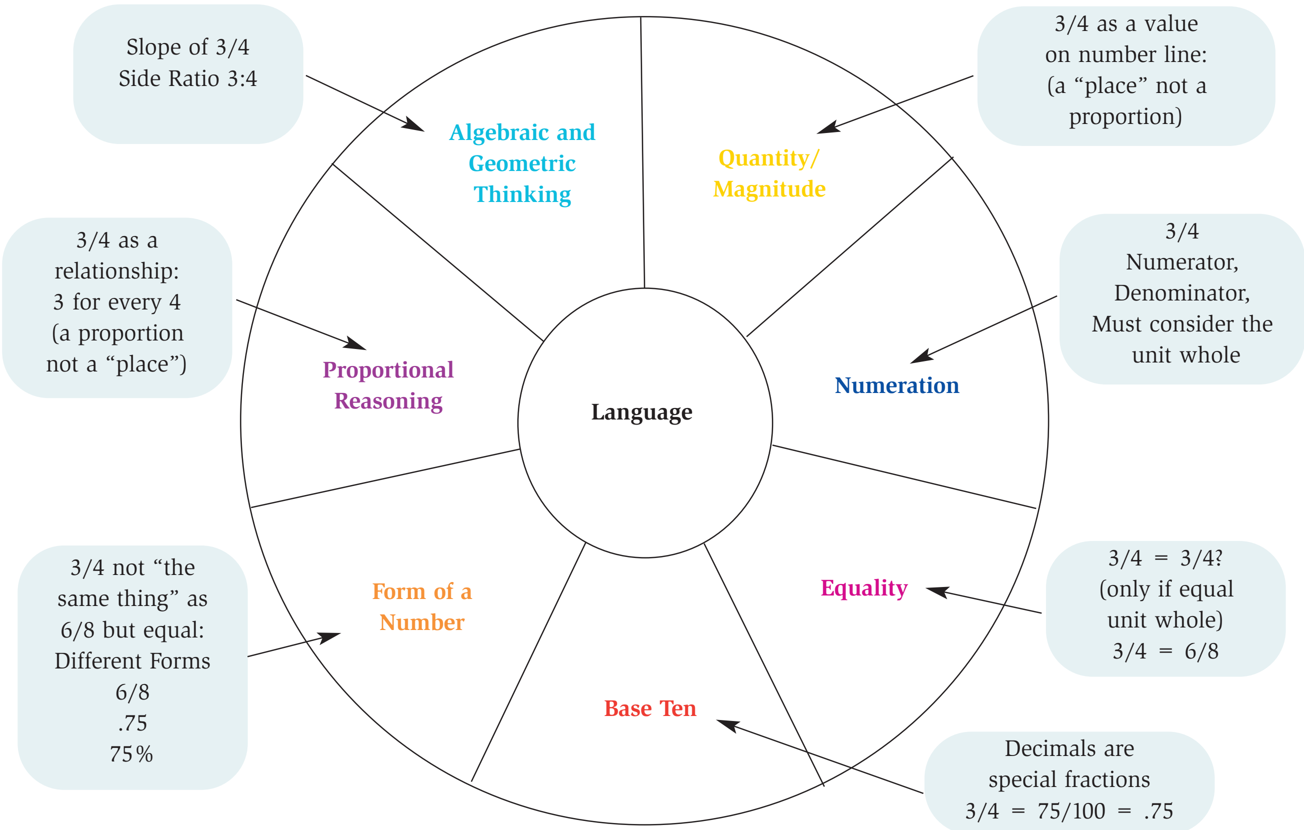
By invoking the habit of considering, in this case, base 10, equality, forms of a number, numeration, quantity, and proportional reasoning, this teacher substantially changed the quality and impact of her lesson.

The following week this teacher was again excited during class. She said that not one student had forgotten to "move the decimal" on the test. This was usually a very common error. But now her students actually understood numbering, had developed some number sense for this topic, and were therefore capable of checking their own work as they negotiated the mathematics on the test.

An understanding of base 10 is another valuable tool in successful estimation, particularly for very large and very small numbers. Finally, a strong estimator will know how to enumerate an estimate, write the estimate in many different forms, and be able to graph or model this estimate using algebraic or geometric reasoning. Like the modular brain emerging from the research, this model respects that these component parts must be linked consistently and discussed repeatedly so that connections are made between these parts to achieve strong mathematical thinking and the acquisition of number sense.

Understanding and utilizing fractions is also a critical skill for students (Ball, 2008). When teaching fractions, consider how they fit into the components of number sense: fractions are not just an algorithm to be taught! (See Figure 2). By considering how fractions fit into each component, the teacher is supported in making mathematical connections first for herself and then for her students (see box, How the Components of Number Sense Affected One Middle School Math Teacher").

Figure 2. The Components of Number Sense and Fractions



Note. V. Faulkner, 2008. From *The Components of Number Sense* by C. Cain, M. Doggett, V. Faulkner, and C. Hale. Raleigh, NC: NC Math Foundations Training, Exceptional Children's Division of the The North Carolina Department of Public Instruction (NCDPI). Copyright 2007. Adapted with permission.

The Components of Number Sense provides a framework for teachers to think of math as a set of connected principles and to present the math to students in this fashion. Whether you are a special educator or a general educator, we hope that you find the Components of Number Sense helpful as you think through what to emphasize with students in your daily mathematics lessons.

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